

L^p ESTIMATES FOR A SINGULAR ENTANGLED QUADRILINEAR FORM

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ABSTRACT. We prove L^p estimates for a continuous version of a dyadic quadrilinear form introduced by Kovač in [6]. This improves the range of exponents from the prequel [3] of the present paper.

1. INTRODUCTION

This article is a continuation of [3]. We are concerned with a quadrilinear singular integral form involving the *entangled* product of four functions on \mathbb{R}^2

$$\mathbf{F}(F_1, F_2, F_3, F_4)(x, y, x', y') := F_1(x, y)F_2(x', y)F_3(x', y')F_4(x, y').$$

For Schwartz functions $F_j \in \mathcal{S}(\mathbb{R}^2)$, the form is given by

$$\Lambda(F_1, F_2, F_3, F_4) := \int_{\mathbb{R}^2} \widehat{\mathbf{F}}(\xi, \eta, -\xi, -\eta) m(\xi, \eta) d\xi d\eta,$$

where $\mathbf{F} := \mathbf{F}(F_1, F_2, F_3, F_4)$ and m is a bounded function on \mathbb{R}^2 , smooth away from the origin. For all multi-indices α up to some large finite order it satisfies¹

$$|\partial^\alpha m(\xi, \eta)| \lesssim (|\xi| + |\eta|)^{-|\alpha|}.$$

In [3] it is shown that

$$|\Lambda(F_1, F_2, F_3, F_4)| \lesssim \|F_1\|_{L^4(\mathbb{R}^2)} \|F_2\|_{L^4(\mathbb{R}^2)} \|F_3\|_{L^4(\mathbb{R}^2)} \|F_4\|_{L^4(\mathbb{R}^2)}. \quad (1.1)$$

Our present goal is to prove L^p estimates for Λ in a larger range of exponents.

Theorem 1. *For $F_1, F_2, F_3, F_4 \in \mathcal{S}(\mathbb{R}^2)$, the quadrilinear form Λ satisfies*

$$|\Lambda(F_1, F_2, F_3, F_4)| \lesssim_{(p_j)} \|F_1\|_{L^{p_1}(\mathbb{R}^2)} \|F_2\|_{L^{p_2}(\mathbb{R}^2)} \|F_3\|_{L^{p_3}(\mathbb{R}^2)} \|F_4\|_{L^{p_4}(\mathbb{R}^2)}$$

whenever $\sum_{j=1}^4 \frac{1}{p_j} = 1$ and $2 < p_j \leq \infty$ for all j .

This theorem is a consequence of the restricted type estimates given by Theorem 3 below. By the decomposition performed in [3], it suffices to prove Theorem 1 for m reduced to a single cone in the frequency plane (ξ, η) . More precisely, it is enough to consider the form

$$\int_0^\infty \mu_t \int_{\mathbb{R}^2} \widehat{\mathbf{F}}(\xi, \eta, -\xi, -\eta) \widehat{\varphi^{(u)}}(t\xi) \widehat{\psi^{(v)}}(t\eta) \widehat{\varphi^{(-u)}}(-t\xi) \widehat{\psi^{(-v)}}(-t\eta) d\xi d\eta \frac{dt}{t} \quad (1.2)$$

where $\varphi^{(u)}(x) = (1 + |u|)^{-25} \varphi(x - u)$ and $\psi^{(v)}(x) = (1 + |v|)^{-10} \psi(x - v)$. The functions $\varphi, \psi \in \mathcal{S}(\mathbb{R})$ are real-valued and ψ is such that $(\int_\eta^\infty |\widehat{\psi}(\tau)|^2 d\tau / \tau)^{1/2}$ belongs to $\mathcal{S}(\mathbb{R})$, $u, v \in \mathbb{R}$ and μ_t are measurable coefficients with $|\mu_t| \leq 1$. We remark that the decomposition

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¹We write $A \lesssim B$ if there is an absolute constant $C > 0$ such that $A \leq CB$. If P depends on a set of parameters P , we write $A \lesssim_P B$. We write $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$.

is not explicitly stated in this manner in [3], but it follows by a minor rephrasing of the arguments. The estimate for (1.2) will be uniform in the parameters u, v .

Since the integral of the Fourier transform of a Schwartz function over a hyperplane in \mathbb{R}^4 equals the integral of the function itself over the perpendicular hyperplane, we can express the form (1.2) as

$$\int_0^\infty \mu_t \int_{\mathbb{R}^2} \mathbf{F} * [\varphi^{(u)} \otimes \psi^{(v)} \otimes \varphi^{(-u)} \otimes \psi^{(-v)}]_t(p, q, p, q) dp dq \frac{dt}{t},$$

where $(f_1 \otimes \dots \otimes f_n)(x_1, \dots, x_n) := f_1(x_1) \dots f_n(x_n)$ and $[f]_t(x_1, \dots, x_n) := t^{-n} f(t^{-1}x)$. We truncate in the scale t , that is, for $N > 0$ we consider $\Lambda_{\varphi, \psi}^N = \Lambda_{\varphi, \psi, \mu, u, v}^N$ given by

$$\Lambda_{\varphi, \psi}^N(F_1, F_2, F_3, F_4) := \int_{2^{-N}}^{2^N} \mu_t \int_{\mathbb{R}^2} \mathbf{F} * [\varphi^{(u)} \otimes \psi^{(v)} \otimes \varphi^{(-u)} \otimes \psi^{(-v)}]_t(p, q, p, q) dp dq \frac{dt}{t},$$

which is well defined for bounded measurable functions F_j with finite measure support. We have the following analogue of Theorem 1 for $\Lambda_{\varphi, \psi}^N$.

Theorem 2. *For bounded measurable functions F_1, F_2, F_3, F_4 with finite measure support, the quadrilinear form $\Lambda_{\varphi, \psi}^N$ satisfies the estimate*

$$|\Lambda_{\varphi, \psi}^N(F_1, F_2, F_3, F_4)| \lesssim_{(p_j)} \|F_1\|_{L^{p_1}(\mathbb{R}^2)} \|F_2\|_{L^{p_2}(\mathbb{R}^2)} \|F_3\|_{L^{p_3}(\mathbb{R}^2)} \|F_4\|_{L^{p_4}(\mathbb{R}^2)} \quad (1.3)$$

whenever $\sum_{j=1}^4 \frac{1}{p_j} = 1$ and $2 < p_j \leq \infty$ for all j .

The bound (1.3) is independent of N, u, v . Approximating $F_j \in \mathcal{S}$ in L^{p_j} with smooth compactly supported functions, Theorem 2 then implies Theorem 1. By the multilinear interpolation and the restricted type theory discussed in [10], Theorem 2 is a consequence of the following (generalized) restricted type estimates.

Theorem 3. *For $j = 1, 2, 3, 4$, let $E_j \subseteq \mathbb{R}^2$ be a set of finite measure. Let k be the largest index such that $|E_k|$ is maximal among the $|E_j|$. Then there exists a subset $E'_k \subseteq E_k$ with $2|E'_k| \geq |E_k|$, such that for any four measurable functions F_j with² $|F_j| \leq \mathbf{1}_{E_j}$ for all j and $|F_k| \leq \mathbf{1}_{E'_k}$ we have the estimate*

$$|\Lambda_{\varphi, \psi}^N(F_1, F_2, F_3, F_4)| \lesssim |E_1|^{\alpha_1} |E_2|^{\alpha_2} |E_3|^{\alpha_3} |E_4|^{\alpha_4}$$

whenever $\sum_{j=1}^4 \alpha_j = 1$ and $-1/2 \leq \alpha_j \leq 1/2$ for all j .

Negative exponents α_j correspond to quasi-Banach space estimates for the dual operators of $\Lambda_{\varphi, \psi}^N$, for which one may consult [10].

Assuming Theorem 1, we now mention how to extend Λ to a bounded operator on $L^{p_1} \times L^{p_2} \times L^{p_3} \times L^{p_4}$ whenever p_j are as in Theorem 1. If $p_j < \infty$ for all j , this follows by density of \mathcal{S} in L^{p_j} . If $p_j = \infty$ for some j , we argue by duality. Note that have at most one exponent equal to ∞ . We sketch the argument when $p_4 = \infty$, the other instances following by symmetry of the form. We know that there is an operator T mapping $L^4 \times L^4 \times L^4$ to $L^{4/3}$ such that

$$\Lambda(F_1, F_2, F_3, F_4) = \int T(F_1, F_2, F_3) F_4.$$

We claim that for $F_j \in \mathcal{S}$, $\|T(F_1, F_2, F_3)\|_{L^1} \lesssim \|F_1\|_{L^{p_1}} \|F_2\|_{L^{p_2}} \|F_3\|_{L^{p_3}}$. Then Λ can be defined on $\mathcal{S} \times \mathcal{S} \times \mathcal{S} \times L^\infty$ and density arguments yield a bounded extension on $L^{p_1} \times L^{p_2} \times L^{p_3} \times L^\infty$. To see the claim we write

$$\|T(F_1, F_2, F_3)\|_{L^1([-M, M]^2)} = \int T(F_1, F_2, F_3) \vartheta$$

²By $\mathbf{1}_A$ we denote the characteristic function of a set $A \subseteq \mathbb{R}^2$.

where ϑ is a modulation times $\mathbf{1}_{[-M,M]^2}$. Then we approximate ϑ weakly in L^4 with smooth compactly supported functions having L^∞ norms uniformly bounded by 1. Applying Theorem 1 for the tuple (p_1, p_2, p_3, ∞) yields the assertion.

Let us briefly comment on the form Λ . For more extensive motivation we refer to [3]. The instance of Λ which was first considered is the trilinear form³ $\Lambda_1(F_1, F_2, F_3) := \Lambda(F_1, F_2, F_3, 1)$. It was introduced by Demeter and Thiele [2]. This trilinear form can also be seen as a simpler version of the twisted paraproduct proposed by Camil Muscalu and sometimes one refers to it with that name as well.

Boundedness of Λ_1 was established by Kovač [6], who first investigated a dyadic model of Λ for a general function F_4 by an induction on scales type argument. See also [5]. This led to an estimate for a dyadic version of Λ_1 whenever $2 < p_1, p_2, p_3 < \infty$ and $1/p_1 + 1/p_2 + 1/p_3 = 1$. Then Kovač passed to the bound for Λ_1 using the square functions of Jones, Seeger and Wright [4]. Bernicot's fiber-wise Calderón-Zygmund decomposition [1] extended the range of exponents to $1 < p_1, p_3 < \infty$, $2 < p_2 \leq \infty$. The transition to the continuous case and the extension of the exponent range both relied on the special structure arising from $F_4 = 1$.

For the quadrilinear form with a general fourth function, the L^4 estimate (1.1) was derived by adapting the induction of scales technique by Kovač to the continuous setting. In the present article we prove estimates in a larger range of exponents by extending his method to the continuous localized context.

By a classical stopping time argument, Theorem 3 is reduced to estimating entangled forms of the type

$$\int_{\Omega} |\mathbf{F} * [\varphi^{(u)} \otimes \psi^{(v)} \otimes \varphi^{(-u)} \otimes \psi^{(-v)}]_t(p, q, p, q)| dp dq \frac{dt}{t}.$$

Here Ω is a certain local region in the upper half space with "regular" boundary. Controlling such objects with the technique from [6] requires an algebraic telescoping identity. In [3], its derivation relies on an identity involving the Fourier transform. The argument is of global nature and we cannot directly repeat it in the localized setting.

We obtain the desired telescoping element in Proposition 8 in Section 2. To overcome the mentioned difficulty, we first restrict the functions F_j to certain projections of the region Ω . This allows us to discard the spatial localization of the form and proceed in the manner of [3]. The issue in the described process is then in estimating boundary terms, representing differences between local and global objects. This requires certain control of the boundary and is carried out in Lemma 6 and Lemma 7 below. Our approach has been inspired by Muscalu, Tao and Thiele [7].

To conclude we remark that in general we do not know of any arguments which could extend the range of exponents from Theorem 1 to $p_j \leq 2$.

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2. LOCAL TELESOPING

First let us set up some notation. A *dyadic interval* is a interval of the form $[2^k m, 2^k(m+1)]$ for some $k, m \in \mathbb{Z}$. We denote the set of all dyadic intervals by \mathcal{I} and the set of all dyadic intervals of length 2^k by \mathcal{I}_k . A *dyadic square* is the Cartesian product of two dyadic intervals of the same length. For a dyadic square S we denote by $\ell(S)$ its sidelength. We write \mathcal{D} for the set of all dyadic squares and \mathcal{D}_k for the set of all dyadic squares of sidelength

³In [3] we called this form T , not to be interchanged with the dual operator introduced above.

2^k . Each $S \in \mathcal{D}$ is divided into four congruent dyadic squares of half the sidelength, called the *children* of S . Conversely, each square in \mathcal{D} has a unique *parent* in \mathcal{D} . Given any two dyadic squares, either one is contained in the other or they are almost disjoint, by which we mean that their intersection has Lebesgue measure zero.

As in [6], we collect the squares into units called *trees*. A finite collection $\mathcal{T} \subseteq \mathcal{D}$ is called a *tree* if there exists a square $R_{\mathcal{T}} \in \mathcal{T}$ called the *root*, satisfying $S \subseteq R_{\mathcal{T}}$ for every $S \in \mathcal{T}$. A tree is called *convex* if for all S_1, S_2, S_3 we have that $S_1 \subseteq S_2 \subseteq S_3$ and $S_1, S_3 \in \mathcal{T}$ imply $S_2 \in \mathcal{T}$. A *leaf* of \mathcal{T} is a dyadic square which is not contained in \mathcal{T} , but its parent is. We denote the set of leaves of \mathcal{T} by $\mathcal{L}(\mathcal{T})$. Note that the leaves of a convex tree partition its root. We split \mathcal{T} into generations of squares of sidelength 2^k . For this we denote

$$\mathcal{T}_k := \mathcal{T} \cap \mathcal{D}_k \quad \text{and} \quad \mathcal{T}_k^c := \mathcal{D}_k \setminus \mathcal{T}_k.$$

For the union of all squares in \mathcal{T}_k we write

$$T_k := \bigcup_{S \in \mathcal{T}_k} S.$$

Observe that for a convex tree \mathcal{T} we have $T_k \subseteq T_{k'}$ if $k \leq k'$, $T_{k'} \neq \emptyset$. The following lemma measures the "size" of the boundary of T_k . It estimates the cardinality of dyadic points

$$\Delta(\mathcal{T}_k) := \partial T_k \cap (2^k \mathbb{Z} \times 2^k \mathbb{Z}).$$

This is a variant of Lemma 4.8 from [7].

Lemma 4. *For any convex tree \mathcal{T} we have*

$$\sum_{k \in \mathbb{Z}} 2^{2k} \# \Delta(\mathcal{T}_k) \lesssim |R_{\mathcal{T}}|.$$

Proof. It suffices to prove the claim for all dyadic points $(p, q) \in \partial T_k$ such that $[p - 2^k, p] \times [q - 2^k, q] \notin \mathcal{T}_k$. For each such point consider the dyadic square

$$S(p, q, k) := [p - 2^k, p - 2^{k-1}] \times [q - 2^k, q - 2^{k-1}]$$

which has area $2^{2(k-1)}$. We claim that squares of this form are pairwise almost disjoint. This will prove the lemma, as they are contained in $3R_{\mathcal{T}}$.

To see the claim, suppose that $S(p, q, k)$ and $S(p', q', k')$ intersect in a set of positive measure. If $k = k'$, then they must coincide since they are dyadic and of the same scale. So suppose that $k < k'$, hence $S(p, q, k)$ is contained in $S(p', q', k')$. Then the point (p, q) is contained in the interior of $[p' - 2^{k'}, p'] \times [q' - 2^{k'}, q']$, which is disjoint from $T_{k'}$. This shows that $(p, q) \in T_k$ but $(p, q) \notin T_{k'}$, contradicting convexity of \mathcal{T} . \square

With any collection of dyadic squares $\mathcal{C} \subseteq \mathcal{D}$ we associate a region in the upper half space \mathbb{R}_+^3 . The region consists of Whitney boxes associated with $S \in \mathcal{C}$ and is defined by

$$\Omega_{\mathcal{C}} := \bigcup_{S \in \mathcal{C}} S \times \left[\frac{\ell(S)}{2}, \ell(S) \right].$$

The case $\mathcal{C} = \mathcal{T}$ for a convex tree \mathcal{T} is depicted in Figure 1. Observe that $\Omega_{\mathcal{T}} = \bigcup_{k \in \mathbb{Z}} \Omega_{\mathcal{T}_k} = \bigcup_{k \in \mathbb{Z}} T_k \times [2^{k-1}, 2^k]$.

Throughout the text, all two-dimensional functions will be measurable, bounded, with finite measure support and positive. Denote

$$\theta(x, y) := (1 + |(x, y)|^4)^{-1}.$$

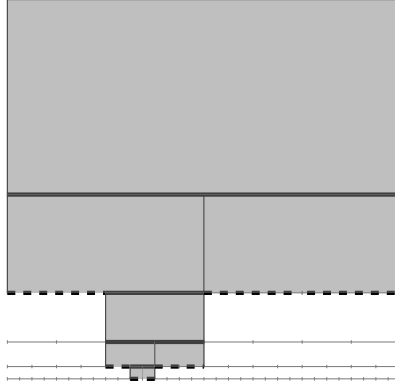


FIGURE 1. Projection of $\Omega_{\mathcal{T}}$ on \mathbb{R}^2_+ . The bold lines represent $S \times \ell(S)$ for $S \in \mathcal{T}$, while the dotted lines correspond to $S \in \mathcal{L}(\mathcal{T})$.

For a function F on \mathbb{R}^2 and $\mathcal{C} \subseteq \mathcal{D}$ we define

$$M(F, \mathcal{C}) := \sup_{(p, q, t) \in \Omega_{\mathcal{C}}} (F^2 * [\theta]_t(p, q))^{1/2}.$$

Denote also

$$\vartheta(x) := (1 + |x|)^{-4}.$$

Now we consider a continuous variant of the Gowers box inner product used in [6]. The following estimate joins a version of the box Cauchy-Schwarz inequality and an estimate of the Gowers box norm by an L²-type average. This is the reason for the restricted range of exponents in Theorem 1.

Lemma 5. *For $(p, q, t) \in \Omega_{\mathcal{C}}$ we have*

$$\mathbf{F} * [\vartheta \otimes \vartheta \otimes \vartheta \otimes \vartheta]_t(p, q, p, q) \leq \prod_{j=1}^4 M(F_j, \mathcal{C}). \quad (2.1)$$

Proof. Denote the left-hand side of (2.1) by $A^{(p, q, t)}(F_1, F_2, F_3, F_4)$ and rewrite it as

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} F_1(x, y) F_2(x', y) [\vartheta]_t(q - y) dy \right) \left(\int_{\mathbb{R}} F_3(x', y') F_4(x, y') [\vartheta]_t(q - y') dy' \right) [\vartheta]_t(p - x) [\vartheta]_t(p - x') dx dx' \quad (2.2)$$

Now we apply the Cauchy-Schwarz inequality with respect to $[\vartheta]_t(p - x) dx$, $[\vartheta]_t(p - x') dx'$, which bounds this term by

$$A^{(p, q, t)}(F_1, F_2, F_2, F_1)^{1/2} A^{(p, q, t)}(F_4, F_3, F_3, F_4)^{1/2}.$$

By symmetry in (p, q) it follows that

$$A^{(p, q, t)}(F_1, F_2, F_2, F_1) \leq A^{(p, q, t)}(F_1, F_1, F_1, F_1)^{1/2} A^{(p, q, t)}(F_2, F_2, F_2, F_2)^{1/2}.$$

Now we write $A^{(p, q, t)}(F_j, F_j, F_j, F_j)$ in the same way as in (2.2) and apply the Cauchy-Schwarz inequality with respect to dy, dy' . This yields

$$A^{(p, q, t)}(F_j, F_j, F_j, F_j) \leq (F_j^2 * [\vartheta \otimes \vartheta]_t(p, q))^2 \leq (F_j^2 * [\theta]_t(p, q))^2,$$

which proves the claim. \square

With functions $\phi_j \in L^1(\mathbb{R})$, $j = 1, 2, 3, 4$, and $\mathcal{C} \subseteq \mathcal{D}$ we associate the local form

$$\Theta_{\phi_1, \phi_2, \phi_3, \phi_4}^{\mathcal{C}}(F_1, F_2, F_3, F_4) := \int_{\Omega_{\mathcal{C}}} \mathbf{F} * [\phi_1 \otimes \phi_2 \otimes \phi_3 \otimes \phi_4]_t(p, q, p, q) dp dq \frac{dt}{t}.$$

To shorten the notation we write $\Theta_{\phi_1, \phi_3}^{\mathcal{C}} := \Theta_{\phi_1, \phi_3, \phi_1, \phi_3}^{\mathcal{C}}$.

The following two complementary lemmas will be used to control error and boundary terms in Proposition 8.

Lemma 6. *For a convex tree \mathcal{T} we have*

$$\sum_{k \in \mathbb{Z}} \Theta_{\vartheta^2, \vartheta^2}^{\mathcal{T}_k}(F_1 \mathbf{1}_{T_k^c}, F_2, F_3, F_4) \lesssim |R_{\mathcal{T}}| \prod_{j=1}^4 M(F_j, \mathcal{T}). \quad (2.3)$$

Observe that by symmetry of (2.3), the same result holds under any permutation of the arguments $F_1 \mathbf{1}_{T_k^c}, F_2, F_3, F_4$.

Proof. For $k \in \mathbb{Z}$ and $t \in [2^{k-1}, 2^k]$ we consider

$$\begin{aligned} & \int_{T_k} \int_{\mathbb{R}^4} \mathbf{F}(F_1 \mathbf{1}_{T_k^c}, F_2, F_3, F_4)(x, y, x', y') [\vartheta \otimes \vartheta \otimes \vartheta \otimes \vartheta]_t(p - x, q - y, p - x', q - y') \\ & \quad \vartheta \otimes \vartheta(t^{-1}(p - x, q - y)) \vartheta \otimes \vartheta(t^{-1}(p - x', q - y')) dx dy dx' dy' dp dq. \end{aligned} \quad (2.4)$$

Note that (2.3) is obtained by integrating this term in $t \in [2^{k-1}, 2^k]$ and summing over $k \in \mathbb{Z}$. We claim that for $(x, y) \in T_k^c$ and $(p, q) \in T_k$ there is a point (a, b) contained in

$$B(p, q) := \{(p', q') \in \partial T_k : p' = p \text{ or } q' = q\} \cup \Delta(\mathcal{T}_k) \quad (2.5)$$

such that $|(p, q) - (x, y)| \geq |(p, q) - (a, b)|$.

This can be seen as follows. By E we denote the intersection of ∂T_k and the line segment between (p, q) and (x, y) . If E contains dyadic points from $\Delta(\mathcal{T}_k)$, we may set (a, b) to be any of these points. Otherwise, E must contain a point of the form $(p', q' + \alpha)$ or $(p' + \alpha, q')$ for some $p', q' \in 2^k \mathbb{Z}$, $\alpha \in (0, 2^k)$. Assume it contains at least one of the form $(p', q' + \alpha)$. For definiteness pick the one with the least distance to (p, q) . In case $q' < q < q' + 2^k$ we know that $(p', q) \in \partial T_k$ and we set $(a, b) = (p', q)$. If $q < q'$, we set $(a, b) = (p', q') \in \Delta(\mathcal{T}_k)$. In case $q > q' + 2^k$ we choose $(p', q' + 2^k) \in \Delta(\mathcal{T}_k)$. Analogously we proceed in the remaining case, that is, if E consists only of points $(p' + \alpha, q')$.

Since $\vartheta \otimes \vartheta \leq \theta$ and θ is radially decreasing, we have for $(p, q), (x, y), (a, b)$ as above

$$\vartheta \otimes \vartheta(t^{-1}(p - x, q - y)) \leq \theta(t^{-1}(p - a, q - b)) \leq \sum_{(a, b) \in B(p, q)} \theta(t^{-1}(p - a, q - b)).$$

Estimating $\vartheta \otimes \vartheta(t^{-1}(p - x', q - y')) \leq 1$, the term (2.4) is bounded by

$$\int_{T_k} \mathbf{F}(F_1 \mathbf{1}_{T_k^c}, F_2, F_3, F_4) * [\vartheta \otimes \vartheta \otimes \vartheta \otimes \vartheta]_t(p, q, p, q) \sum_{(a, b) \in B(p, q)} \theta(t^{-1}(p - a, q - b)) dp dq.$$

Applying Lemma 5, the last display is no greater than

$$\left(M(F_1 \mathbf{1}_{T_k^c}, \mathcal{T}_k) \prod_{j=2}^4 M(F_j, \mathcal{T}_k) \right) \int_{T_k} \sum_{(a, b) \in B(p, q)} \theta(t^{-1}(p - a, q - b)) dp dq.$$

Observe that by homogeneity of the inequality (2.3) we may assume $M(F_j, \mathcal{T}) = 1$ for all j . Due to this fact and by symmetry in p, q , it suffices to further estimate

$$\sum_{Q \in \mathcal{I}_k} \int_Q \sum_{a: \{a\} \times Q \subseteq \partial T_k} \int_{\mathbb{R}} \theta(t^{-1}(p-a, 0)) dp dq + \sum_{(a,b) \in \Delta(\mathcal{T}_k)} \int_{\mathbb{R}^2} \theta(t^{-1}(p-a, q-b)) dp dq.$$

Integrating the function θ , the last display is estimated by a constant times

$$\sum_{Q \in \mathcal{I}_k} \int_Q t \# \{a : \{a\} \times Q \subseteq \partial T_k\} + t^2 \# \Delta(\mathcal{T}_k) \lesssim 2^{2k} \# \Delta(\mathcal{T}_k).$$

Therefore, up to a constant, (2.3) is bounded by

$$\sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} 2^{2k} \# \Delta(\mathcal{T}_k) \frac{dt}{t} \lesssim \sum_{k \in \mathbb{Z}} 2^{2k} \# \Delta(\mathcal{T}_k) \lesssim |\mathcal{R}_{\mathcal{T}}|,$$

which is the desired result in view of the normalization $M(F_j, \mathcal{T}) = 1$. The last inequality follows from Lemma 4. \square

Lemma 7. *For a convex tree \mathcal{T} we have*

$$\sum_{k \in \mathbb{Z}} \Theta_{\partial^2, \partial^2}^{\mathcal{T}_k^c} (F_1 \mathbf{1}_{T_k}, F_2 \mathbf{1}_{T_k}, F_3 \mathbf{1}_{T_k}, F_4 \mathbf{1}_{T_k}) \lesssim |\mathcal{R}_{\mathcal{T}}| \prod_{j=1}^4 M(F_j, \mathcal{T}). \quad (2.6)$$

Proof. Proceeding in the exact same way as in the proof of Lemma 6 we see that the left-hand side of (2.6) is bounded by

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \left(\prod_{j=1}^4 M(F_j \mathbf{1}_{T_k}, \mathcal{T}_k^c) \right) \int_{T_k^c} \sum_{(a,b) \in B(p,q)} \theta(t^{-1}(p-a, q-b)) dp dq \\ & \lesssim \sum_{k \in \mathbb{Z}} \left(\prod_{j=1}^4 M(F_j \mathbf{1}_{T_k}, \mathcal{T}_k^c) \right) 2^{2k} \# \Delta(\mathcal{T}_k), \end{aligned}$$

where $B(p, q)$ is defined as in (2.5). We claim that for each j we have

$$M(F_j \mathbf{1}_{T_k}, \mathcal{T}_k^c) \lesssim M(F_j, \mathcal{T}).$$

Together with an application of Lemma 4 this will finish the proof.

The claim can be rephrased as follows: for each $(p, q) \in T_k^c$ we have

$$(F_j^2 \mathbf{1}_{T_k} * [\theta]_t(p, q))^{1/2} \lesssim M(F_j, \mathcal{T}).$$

First we set $(p, q) = 0$ without loss of generality. Also, we may assume that T_k is contained in the quadrant $\{(p, q) : p \geq 0, q \geq 0\}$, as otherwise we restrict T_k to each of the four quadrants and all parts are treated in the same way. Denote

$$r := \min_{(a,b) \in \partial T_k} |(a, b)|.$$

Take any point (a, b) which minimizes the distance and consider the closed cone C in \mathbb{R}^2 with vertex 0 and aperture $\pi/2$, its axis being the line spanned by (a, b) . Observe that each $(x, y) \in T_k \cap C$ satisfies $|(x, y)| \geq |(x, y) - (a, b)|$ and thus $\theta(x, y) \leq \theta(x-a, y-b)$. If $T_k \setminus C \neq \emptyset$, then we iterate with T_k replaced by $T_k \setminus C$. We find a point $(a', b') \in \partial T_k \cap \partial(T_k \setminus C)$ and a cone C' such that for each $(x, y) \in (T_k \setminus C) \cap C'$ we have $|(x, y)| \geq |(x, y) - (a', b')|$ and so $\theta(x, y) \leq \theta(x-a', y-b')$. Since $C \cup C'$ covers T_k , for each $(x, y) \in T_k$ we have

$$\theta(x, y) \leq \theta(a-x, b-y) + \theta(a'-x, b'-y).$$

Therefore,

$$(F_j^2 \mathbf{1}_{T_k} * [\theta]_t(0))^{1/2} \lesssim \left(\sup_{(a,b,t) \in \Omega_{T_k}} F_j^2 \mathbf{1}_{T_k} * [\theta]_t(a,b) \right)^{1/2} \leq \sup_{(a,b,t) \in \Omega_{\mathcal{T}}} (F_j^2 * [\theta]_t(a,b))^{1/2}$$

as desired. \square

For a function $f \in \mathcal{S}(\mathbb{R})$ we consider the Schwartz seminorm

$$\|f\| := \sup_{x \in \mathbb{R}} (1 + |x|)^8 |f(x)| + (1 + |x|)^9 |f'(x)|.$$

Now we are ready to state the estimate which will take the place of the telescoping identities used in [6], [3].

Proposition 8. *Let (ρ_i, σ_i) be two pairs of real-valued Schwartz functions which satisfy*

$$-t\partial_t |\widehat{\rho}_i(t\tau)|^2 = |\widehat{\sigma}_i(t\tau)|^2. \quad (2.7)$$

Then we have for any convex tree \mathcal{T}

$$\Theta_{\rho_1, \sigma_2}^{\mathcal{T}}(F_1, F_2, F_3, F_4) + \Theta_{\sigma_1, \rho_2}^{\mathcal{T}}(F_1, F_2, F_3, F_4) \lesssim_c |R_{\mathcal{T}}| \prod_{j=1}^4 M(F_j, \mathcal{T}), \quad (2.8)$$

where $c = \|\rho_1\|^2 \|\sigma_2\|^2 + \|\sigma_1\|^2 \|\rho_2\|^2 + \|\rho_1\|^2 \|\rho_2\|^2$.

Proof. By homogeneity of (2.8) we may assume $M(F_j, \mathcal{T}) = 1$ for all j . By scaling invariance we may suppose $|R_{\mathcal{T}}| = 1$. Thus, we are set to establish

$$\Theta_{\rho_1, \sigma_2}^{\mathcal{T}}(F_1, F_2, F_3, F_4) + \Theta_{\sigma_1, \rho_2}^{\mathcal{T}}(F_1, F_2, F_3, F_4) \lesssim_c 1. \quad (2.9)$$

Denote $\Psi := \rho_1 \otimes \rho_2 \otimes \rho_1 \otimes \rho_2$. By the fundamental theorem of calculus we have

$$[\Psi]_{2^{k-1}} - [\Psi]_{2^k} = \int_{2^{k-1}}^{2^k} (-t\partial_t [\Psi]_t) \frac{dt}{t}. \quad (2.10)$$

We convolve the equality (2.10) with \mathbf{F} and evaluate the convolution at (p, q, p, q) . Then we integrate in (p, q) over T_k and sum over $k \in \mathbb{Z}$. Writing T_k as the almost disjoint union of $S \in \mathcal{T}_k$, the left-hand side of (2.10) becomes

$$L := \sum_{k \in \mathbb{Z}} \sum_{S \in \mathcal{T}_k} \left(\sum_{S' \text{ child of } S} \int_{S'} \mathbf{F} * [\Psi]_{\ell(S')}(p, q, p, q) dp dq - \int_S \mathbf{F} * [\Psi]_{\ell(S)}(p, q, p, q) dp dq \right)$$

Since \mathcal{T} is convex, each square $S \in \mathcal{T} \setminus \{R_{\mathcal{T}}\}$ has all four children S' in $\mathcal{T} \cup \mathcal{L}(\mathcal{T})$. Thus, the last display is a telescoping sum which equals

$$\sum_{S \in \mathcal{L}(\mathcal{T})} \int_S \mathbf{F} * [\Psi]_{\ell(S)}(p, q, p, q) dp dq - \int_{R_{\mathcal{T}}} \mathbf{F} * [\Psi]_{\ell(R_{\mathcal{T}})}(p, q, p, q) dp dq.$$

We bound $|\Psi| \lesssim_c \vartheta^2 \otimes \vartheta^2 \otimes \vartheta^2 \otimes \vartheta^2$ and apply Lemma 5. This yields

$$|L| \lesssim_c \left(\sum_{S \in \mathcal{L}(\mathcal{T})} |S| + 1 \right) \lesssim 1.$$

The last estimate follows since the leaves of \mathcal{T} partition the root $R_{\mathcal{T}}$.

Now we consider the right-hand side of (2.10), which after convolving it with \mathbf{F} , integrating over T_k and summing in $k \in \mathbb{Z}$ results in

$$R := \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \int_{T_k} \mathbf{F}((F_j)_{j \in J}) * (-t\partial_t [\Psi]_t)(p, q, p, q) dp dq \frac{dt}{t},$$

where $J := \{1, 2, 3, 4\}$. First we show that up to a controllable error, we may suppose that the functions F_j are supported on T_k . For $j \in J$ we write $F_j = F_j \mathbf{1}_{T_k} + F_j \mathbf{1}_{T_k^c}$. Then

$$R = M + E,$$

where the main term is defined as

$$M := \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \int_{T_k} \mathbf{F}((F_j \mathbf{1}_{T_k})_{j \in J}) * (-t \partial_t [\Psi]_t)(p, q, p, q) dp dq \frac{dt}{t}$$

and the error term equals

$$E := \sum_{((X_{j,k})_{k \in \mathbb{Z}})_{j \in J}} \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \int_{T_k} \mathbf{F}((F_j \mathbf{1}_{X_{j,k}})_{j \in J}) * (-t \partial_t [\Psi]_t)(p, q, p, q) dp dq \frac{dt}{t},$$

where the outer summation is over $((X_{j,k})_{k \in \mathbb{Z}})_{j \in J} \in \{T, T^c\}^4 \setminus \{(T, T, T, T)\}$ for $T := (T_k)_{k \in \mathbb{Z}}$, $T^c := (T_k^c)_{k \in \mathbb{Z}}$.

To treat E we expand $-t \partial_t [\Psi]_t = -t \partial_t ([\rho_1]_t \otimes [\rho_2]_t \otimes [\rho_1]_t \otimes [\rho_2]_t)$ and use the chain rule, which results in four terms. By symmetry we consider only $-t \partial_t ([\rho_1]_t \otimes [\rho_2]_t \otimes [\rho_1]_t \otimes [\rho_2]_t)$, on which we use the identity

$$-t \partial_t [\rho_1]_t = -t \partial_t \left(\frac{1}{t} \rho_1 \left(\frac{x}{t} \right) \right) = \frac{1}{t} \rho_1 \left(\frac{x}{t} \right) + \frac{1}{t} \frac{x}{t} \rho_1' \left(\frac{x}{t} \right). \quad (2.11)$$

and bound the right-hand side of (2.11) by $\lesssim_c [\vartheta^2]_t$. This gives $|t \partial_t [\Psi]_t| \lesssim_c [\vartheta^2 \otimes \vartheta^2 \otimes \vartheta^2 \otimes \vartheta^2]_t$. By Lemma 6 we then have $|E| \lesssim_c 1$.

To estimate M we expand the convolution and interchange the order of integration such that the integration in (p, q) becomes the innermost. For now we consider only this innermost integral, which we write in the form

$$\int_{T_k} -t \partial_t \left(\left([\rho_1]_t(p-x) [\rho_1]_t(p-x') \right) \left([\rho_2]_t(q-y) [\rho_2]_t(q-y') \right) \right) dp dq.$$

Deriving the product of $[\rho_1]_t(p-x) [\rho_1]_t(p-x')$ and $[\rho_2]_t(q-y) [\rho_2]_t(q-y')$ yields two terms. Using Fubini and moving the derivative outside the integral we arrive at

$$\sum_{Q \in \mathcal{I}_k} \left(-t \partial_t \int_{T_{Q,1}} [\rho_1]_t(p-x) [\rho_1]_t(p-x') dp \right) \int_Q [\rho_2]_t(q-y) [\rho_2]_t(q-y') dq \quad (2.12)$$

$$+ \sum_{P \in \mathcal{I}_k} \int_P [\rho_1]_t(p-x) [\rho_1]_t(p-x') dp \left(-t \partial_t \int_{T_{P,2}} [\rho_2]_t(q-y) [\rho_2]_t(q-y') dq \right), \quad (2.13)$$

where for a dyadic interval Q we denote $T_{Q,1} := \cup_{P: P \times Q \in \mathcal{T}} P$ and $T_{P,2}$ is defined analogously. As both parts are treated in the same way, we further investigate only (2.12).

The identity (2.7) implies

$$-t \partial_t \int_{\mathbb{R}} [\rho_1]_t(p-x) [\rho_1]_t(p-x') dp = \int_{\mathbb{R}} [\sigma_1]_t(p-x) [\sigma_1]_t(p-x') dp,$$

which can be seen by an application of the inverse Fourier transform on (2.7). Hence,

$$-t \partial_t \int_{T_{Q,1}} [\rho_1]_t(p-x) [\rho_1]_t(p-x') dp = \int_{T_{Q,1}} [\sigma_1]_t(p-x) [\sigma_1]_t(p-x') dp + b_1,$$

where b_1 is the boundary portion

$$b_1 := \int_{\mathbb{R} \setminus T_{Q,1}} [\sigma_1]_t(p-x) [\sigma_1]_t(p-x') dp + t \partial_t \int_{\mathbb{R} \setminus T_{Q,1}} [\rho_1]_t(p-x) [\rho_1]_t(p-x') dp.$$

Therefore we have

$$M = \left(\sum_{k \in \mathbb{Z}} \Theta_{\sigma_1, \rho_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) + \Theta_{\rho_1, \sigma_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) \right) + B_1 + B_2, \quad (2.14)$$

where the boundary term B_1 emerges from b_1 and equals

$$B_1 := \sum_{k \in \mathbb{Z}} \int_{2^{k-1}}^{2^k} \sum_{Q \in \mathcal{I}_k} \int_Q \int_{\mathbb{R} \setminus T_{Q,1}} \mathbf{F}((F_j \mathbf{1}_{T_k})_{j \in J})(x, y, x', y') \\ \left([\sigma_1]_t(p-x)[\sigma_1]_t(p-x') + t \partial_t([\rho_1]_t(p-x)[\rho_1]_t(p-x')) \right) [\rho_2]_t(q-x)[\rho_2]_t(q-x') \\ dx dy dx' dy' dp dq \frac{dt}{t}.$$

The boundary term B_2 arises from the treatment of (2.13) and is analogous to B_1 with (σ_1, ρ_2) replaced by (ρ_1, σ_2) . For B_1, B_2 we derive by t using (2.11) and dominate the resulting functions by $\lesssim_c \vartheta^2$. Note that

$$|B_1 + B_2| \lesssim_c \sum_{k \in \mathbb{Z}} \Theta_{\vartheta^2, \vartheta^2}^{\mathcal{T}_k^c}((F_j \mathbf{1}_{T_k})_{j \in J}) \lesssim 1,$$

where the last inequality follows by Lemma 7.

Summarizing, since $L = R = M + E$, using (2.14) yields the identity

$$L = \left(\sum_{k \in \mathbb{Z}} \Theta_{\sigma_1, \rho_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) + \Theta_{\rho_1, \sigma_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) \right) + B_1 + B_2 + E. \quad (2.15)$$

Proposition 8 now follows by writing

$$\Theta_{\rho_1, \sigma_2}^{\mathcal{T}}((F_j)_{j \in J}) + \Theta_{\sigma_1, \rho_2}^{\mathcal{T}}((F_j)_{j \in J})$$

in the form

$$\sum_{k \in \mathbb{Z}} \Theta_{\sigma_1, \rho_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) + \Theta_{\rho_1, \sigma_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{T_k})_{j \in J}) \\ + \sum_{((X_{j,k})_{k \in \mathbb{Z}})_{j \in J}} \sum_{k \in \mathbb{Z}} \Theta_{\rho_1, \sigma_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{X_{j,k}})_{j \in J}) + \Theta_{\sigma_1, \rho_2}^{\mathcal{T}_k}((F_j \mathbf{1}_{X_{j,k}})_{j \in J}),$$

where in the second line, the outer sum runs over $((X_{j,k})_{k \in \mathbb{Z}})_{j \in J} \in \{T, T^c\}^4 \setminus \{(T, T, T, T)\}$ for T as above. Using (2.15) together with

$$|L - B_1 - B_2 - E| \lesssim_c 1$$

and evoking Lemma 6 two more times finally yields (2.9). \square

3. TREE ESTIMATE

In this section we derive an estimate for a quadrisublinear variant of $\Lambda_{\varphi, \psi}^N$ restricted to $\Omega_{\mathcal{T}}$ for a convex tree \mathcal{T} . This form is given by

$$\tilde{\Theta}_{\varphi, \psi}^{\mathcal{T}}(F_1, F_2, F_3, F_4) := \int_{\Omega_{\mathcal{T}}} |\mathbf{F} * [\varphi^{(u)} \otimes \psi^{(v)} \otimes \varphi^{(-u)} \otimes \psi^{(-v)}]_t(p, q, p, q)| dp dq \frac{dt}{t}.$$

It can also be recognized as a quadrisublinear version of $\Theta_{\varphi^{(u)}, \psi^{(v)}, \varphi^{(-u)}, \psi^{(-v)}}^{\mathcal{T}}$.

Proposition 9. *We have the estimate*

$$\tilde{\Theta}_{\varphi, \psi}^{\mathcal{T}}(F_1, F_2, F_3, F_4) \lesssim |R_{\mathcal{T}}| \prod_{j=1}^4 M(F_j, \mathcal{T}). \quad (3.1)$$

The proof of Proposition 9 proceeds in a very similar way as the proof of the L⁴ bound (1.1). Besides replacing [3, Lemma 3] with Proposition 8, the only modification is the choice of a faster decaying superposition of the Gaussian exponential functions (3.2). For completeness we summarize all steps of the proof, interested readers are referred to [3].

Proof. By homogeneity and scale-invariance we may suppose $M(F_j, \mathcal{T}) = 1$ and $|R_{\mathcal{T}}| = 1$. First we expand the left-hand side of (3.1) and use the triangle inequality to arrive at

$$\int_{\Omega_{\mathcal{T}}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} F_1(x, y) F_2(x', y) [\psi^{(v)}]_t(q - y) dy \int_{\mathbb{R}} F_3(x', y') F_4(x, y') [\psi^{(-v)}]_t(q - y') dy' \right| \\ \|\varphi^{(u)}\|_t(p - x) \|\varphi^{(-u)}\|_t(p - x') dx dx' dp dq \frac{dt}{t}.$$

By an application of the Cauchy-Schwarz inequality, this is bounded by

$$\Theta_{|\varphi^{(u)}|, \psi^{(v)}, |\varphi^{(-u)}|, \psi^{(v)}}^{\mathcal{T}}(F_1, F_2, F_2, F_1)^{1/2} \Theta_{|\varphi^{(u)}|, \psi^{(-v)}, |\varphi^{(-u)}|, \psi^{(-v)}}^{\mathcal{T}}(F_4, F_3, F_3, F_4)^{1/2}.$$

As both terms are treated analogously, we consider the first one only. We shall now apply the telescoping identity, for which we dominate $\varphi^{(\pm u)}$ with a superposition of Gaussians. Denote the L¹-normalized Gaussian exponential function rescaled by $\alpha > 0$ by

$$g_{\alpha}(x) := \frac{1}{\sqrt{\pi\alpha}} e^{-\left(\frac{x}{\alpha}\right)^2}.$$

Consider the superposition of the functions g_{α} given by

$$\Phi(x) := \int_1^{\infty} \frac{1}{\alpha^{21}} e^{-\left(\frac{x}{\alpha}\right)^2} d\alpha = \frac{1}{\sqrt{\pi}} \int_1^{\infty} \frac{1}{\alpha^{20}} g_{\alpha}(x) d\alpha. \quad (3.2)$$

For large x we have $\Phi(x) \sim x^{20}$, which can be seen by the change of variables $\alpha' = (x/\alpha)^2$ and by inductive integration by parts. The power of α is now larger as in [3], as due to Proposition 8 we need control over higher Schwartz seminorms of g_{α} .

Since $\varphi^{(\pm u)} \in \mathcal{S}(\mathbb{R}^2)$, we can bound it by Φ times a positive constant, which is uniform in u . By positivity of

$$\Theta_{|\varphi^{(u)}|, \psi^{(v)}, |\varphi^{(-u)}|, \psi^{(v)}}^{\mathcal{T}}(F_1, F_2, F_2, F_1) = \int_{\Omega_{\mathcal{T}}} \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} F_1(x, y) F_2(x', y) [\psi^{(v)}]_t(q - y) dy \right)^2 \\ \|\varphi^{(u)}\|_t(p - x) \|\varphi^{(u)}\|_t(p - x') dx dx' dp dq \frac{dt}{t},$$

we can estimate this term up to a constant by

$$\int_1^{\infty} \int_1^{\infty} \Theta_{g_{\alpha}, \psi^{(v)}, g_{\beta}, \psi^{(v)}}^{\mathcal{T}}(F_1, F_2, F_2, F_1) \frac{d\alpha}{\alpha^{20}} \frac{d\beta}{\beta^{20}}.$$

We split the integration into the regions $\alpha \geq \beta$ and $\alpha < \beta$. By symmetry it suffices to estimate the region $\alpha \geq \beta$ only, on which $\beta g_{\beta} \leq \alpha g_{\alpha}$ for $\alpha, \beta \geq 1$. This leaves us with

$$\int_1^{\infty} \Theta_{g_{\alpha}, \psi^{(v)}}^{\mathcal{T}}(F_1, F_2, F_2, F_1) \frac{d\alpha}{\alpha^{19}}.$$

Now we are ready to apply Proposition 8 with $(\rho_1, \sigma_1) = (g_{\alpha}, h_{\alpha})$ and $(\rho_1, \sigma_2) = (\phi, \psi^{(v)})$, where $h_{\alpha}(x) := \alpha(g_{\alpha})'(x)$ and

$$\widehat{\phi}(\xi) := \left(\int_{\xi}^{\infty} |\widehat{\psi^{(v)}}(\tau)|^2 \frac{d\tau}{\tau} \right)^{1/2},$$

which is a Schwartz function by our condition on ψ . Proposition 8 yields

$$\Theta_{g_{\alpha}, \psi^{(v)}}^{\mathcal{T}}(F_1, F_2, F_2, F_1) \lesssim -\Theta_{h_{\alpha}, \phi}^{\mathcal{T}}(F_1, F_2, F_2, F_1) + c \quad (3.3)$$

with $c = \|g_\alpha\|^2 \|\psi^{(v)}\|^2 + \|\phi\|^2 \|h_\alpha\|^2 + \|g_\alpha\|^2 \|\phi\|^2 \lesssim \alpha^{16}$. Thus it remains to estimate the form on the right-hand side of (3.3).

In the second iteration of the procedure we bound $|\Theta_{h_\alpha, \phi}^\mathcal{T}(F_1, F_2, F_2, F_1)|$ by

$$\int_{\Omega_\mathcal{T}} \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}} F_1(x, y) F_1(x, y') [h_\alpha]_t (p - x) dx \int_{\mathbb{R}} F_2(x', y') F_2(x', y) [h_\alpha]_t (p - x') dx' \right| \\ \|\phi\|_t (q - y) \|\phi\|_t (q - y') dy dy' dp dq \frac{dt}{t}.$$

Again we apply the Cauchy-Schwarz inequality and arrive to

$$|\Theta_{h_\alpha, \phi}^\mathcal{T}(F_1, F_2, F_2, F_1)| \leq \Theta_{h_\alpha, |\phi|}^\mathcal{T}(F_1, F_1, F_1, F_1)^{1/2} \Theta_{h_\alpha, |\phi|}^\mathcal{T}(F_2, F_2, F_2, F_2)^{1/2}$$

Dominating the rapidly decaying $|\phi|$ by a positive constant times Φ gives

$$\Theta_{h_\alpha, |\phi|}^\mathcal{T}(F_1, F_1, F_1, F_1) \lesssim \int_1^\infty \int_1^\infty \Theta_{h_\alpha, g_\gamma, h_\alpha, g_\delta}^\mathcal{T}(F_1, F_1, F_1, F_1) \frac{d\gamma}{\gamma^{20}} \frac{d\delta}{\delta^{20}}.$$

As before, by symmetry this reduces to having to estimate

$$\int_1^\infty \Theta_{h_\alpha, g_\gamma}^\mathcal{T}(F_1, F_1, F_1, F_1) \frac{d\gamma}{\gamma^{19}}.$$

Now we apply Proposition 8 to the pairs $(\rho_1, \sigma_1) = (g_\alpha, h_\alpha)$ and $(\rho_2, \sigma_2) = (g_\gamma, h_\gamma)$, giving

$$\Theta_{h_\alpha, g_\gamma}^\mathcal{T}(F_1, F_1, F_1, F_1) \lesssim -\Theta_{g_\alpha, h_\gamma}^\mathcal{T}(F_1, F_1, F_1, F_1) + c$$

with $c = \|g_\alpha\|^2 \|h_\gamma\|^2 + \|g_\gamma\|^2 \|h_\alpha\|^2 + \|g_\alpha\|^2 \|g_\gamma\|^2 \lesssim \alpha^{16} \gamma^{16}$. Finally observe that

$$\Theta_{g_\alpha, h_\gamma}^\mathcal{T}(F_1, F_1, F_1, F_1) \geq 0,$$

which can be seen by writing it as an integral of a square multiplied with $g_\alpha \geq 0$. Thus,

$$\Theta_{h_\alpha, g_\gamma}^\mathcal{T}(F_1, F_1, F_1, F_1) \leq 1.$$

This concludes the proof in view of our normalization. \square

4. COMPLETING THE PROOF OF THEOREM 3

Now we are ready to establish the restricted type estimate from Theorem 3. We adapt the approach of [10] and also rely on [9].

Proof of Theorem 3. First note that by quadrilinearity of $\Lambda_{\varphi, \psi}^N$ it suffices to prove the theorem for positive functions F_j , as otherwise we split them into real and imaginary, positive and negative parts.

For $j = 1, 2, 3, 4$ let α_j be such that $-1/2 \leq \alpha_j \leq 1/2$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$. For each j let $E_j \subseteq \mathbb{R}^2$ be measurable. Without loss of generality we may assume $|E_1|$ is maximal among the $|E_j|$. Note that for $a = 2^k$ we have the scaling identity

$$\Lambda_{\varphi, \psi}^N(F_1, F_2, F_3, F_4) = a^2 \Lambda_{\varphi, \psi}^{N/a}(F_1(a \cdot), F_2(a \cdot), F_3(a \cdot), F_4(a \cdot)).$$

Since our bound will be independent of N , by $\sum_j \alpha_j = 1$ we may then suppose $1 \leq |E_1| \leq 4$. All squares which we consider in this section are assumed to have their side-lengths in the interval $[2^{-N}, 2^N]$.

For F on \mathbb{R}^2 we denote the quadratic Hardy-Littlewood maximal function by

$$\mathcal{M}(F) := \sup_S \left(\frac{1}{|S|} \int_S F^2 \right)^{1/2} \mathbf{1}_S,$$

where the supremum is taken over all (not necessarily dyadic) squares in \mathbb{R}^2 with sides parallel to the coordinate axes. From now on, by the word "average" we will always mean the second power average as in the definition of $\mathcal{M}(F)$. Define the exceptional set

$$H := \bigcup_{j=1}^4 \{ \mathcal{M}(|E_j|^{-1/2} \mathbf{1}_{E_j}) > 2^{10} \}.$$

By the Hardy-Littlewood maximal theorem we have $|H| \leq 1/18$. Let \mathcal{R} be the set of all dyadic squares $R \subseteq H$ which are maximal with respect to set inclusion. Denote by $3R$ the square with the same center as R but with three times the sidelength of R . We set $E'_1 := E_1 \setminus \bigcup_{R \in \mathcal{R}} 3R$. Then $2|E'_1| \geq |E_1|$.

Suppose we are given four functions F_j with $|F_j| \leq \mathbf{1}_{E_j}$ for all j and $|F_1| \leq \mathbf{1}_{E'_1}$. Since $\alpha_j \leq 1/2$ and $|E_1| \leq 4$, it suffices to prove

$$|\Lambda_{\varphi, \psi}^N(F_1, F_2, F_3, F_4)| \lesssim |E_1|^{1/2} |E_2|^{1/2} |E_3|^{1/2} |E_4|^{1/2}.$$

If we set $G_j := |E_j|^{-1/2} F_j$, then the inequality we need to establish reads

$$|\Lambda_{\varphi, \psi}^N(G_1, G_2, G_3, G_4)| \lesssim 1.$$

Observe that $\|G_j\|_{L^2(\mathbb{R}^2)} \leq 1$ for all j .

We split $\mathbb{R}^2 \times [2^{-N}, 2^N]$ into the regions $\Omega_{\{S\}} = S \times [\ell(S)/2, \ell(S)]$, $S \in \mathcal{D}$, and consider the cases $S \subseteq H$ and $S \not\subseteq H$. By the triangle inequality we estimate

$$|\Lambda_{\varphi, \psi}^N| \leq \sum_{S \subseteq H} \tilde{\Theta}_{\varphi, \psi}^{\{S\}} + \sum_{S \not\subseteq H} \tilde{\Theta}_{\varphi, \psi}^{\{S\}}.$$

First we consider the sum over $S \not\subseteq H$. For $k \in \mathbb{Z}$ let \mathcal{S}_k be the set of all dyadic squares S for which

$$2^{k-1} < \max_{j \in \{1, 2, 3, 4\}} \sup_{S' \supseteq S} \left(\frac{1}{|S'|} \int_{S'} G_j^2 \right)^{1/2} \leq 2^k.$$

The supremum is taken over all (not necessarily dyadic) squares $S' \supseteq S$ in \mathbb{R}^2 with sides parallel to the coordinate axes. Denote by \mathcal{R}_k the collection of the maximal squares in \mathcal{S}_k with respect to set inclusion. For $R \in \mathcal{R}_k$ we define

$$\mathcal{T}_R := \{S \in \mathcal{S}_k : S \subseteq R\},$$

which is a convex tree with the root R . Convexity follows from monotonicity of the supremum. By construction, if $S \not\subseteq H$, for each j the average of $|E_j|^{-1/2} \mathbf{1}_{E_j}$ over S is no greater than 2^{10} . Thus, the same holds for the average of G_j over S . Therefore,

$$\{S : S \not\subseteq H\} \subseteq \bigcup_{k \leq 10} \mathcal{S}_k$$

and we can split the summation as

$$\sum_{S \not\subseteq H} \tilde{\Theta}_{\varphi, \psi}^{\{S\}} \leq \sum_{k \leq 10} \sum_{R \in \mathcal{R}_k} \sum_{S \in \mathcal{T}_R} \tilde{\Theta}_{\varphi, \psi}^{\{S\}} = \sum_{k \leq 10} \sum_{R \in \mathcal{R}_k} \tilde{\Theta}_{\varphi, \psi}^{\mathcal{T}_R}.$$

For the forms on the right-hand side we have by Proposition 9 that

$$\tilde{\Theta}_{\varphi, \psi}^{\mathcal{T}_R}(G_1, G_2, G_3, G_4) \lesssim |R| \prod_{j=1}^4 M(G_j, \mathcal{T}_R). \quad (4.1)$$

To estimate the right-hand side of (4.1) we discretize the function θ by a standard approximation with characteristic functions of balls of radius at least 1. We now sketch the required argument. Denote by B_r the ball of radius r centered at 0 in \mathbb{R}^2 . We write

$$G_j^2 * [\theta]_t = G_j^2 * [\theta \mathbf{1}_{B_1}]_t + G_j^2 * [\theta \mathbf{1}_{B_1^c}]_t.$$

Let $(p, q, t) \in S \times [\ell(S)/2, \ell(S)] \subseteq \Omega_{\mathcal{T}_R}$ and assume $(p, q) = 0$. On B_1 we have

$$G_j^2 * [\theta \mathbf{1}_{B_1}]_t(0) \lesssim \|\theta\|_{L^\infty(\mathbb{R}^2)} \frac{1}{(2t)^2} \int_{[-t, t]^2} G_j^2 \lesssim \frac{1}{(2\ell(S))^2} \int_{[-\ell(S), \ell(S)]^2} G_j^2 \lesssim 2^{2k}. \quad (4.2)$$

For the part on B_1^c we consider the function $\theta \mathbf{1}_{B_1^c} + \frac{1}{2} \mathbf{1}_{B_1}$. It dominates $\theta \mathbf{1}_{B_1^c}$, is positive and radially decreasing. Therefore it can be approximated pointwise by a monotonously increasing sequence of simple functions of the form

$$E = \sum_{i=1}^n a_i \mathbf{1}_{B_{r_i}}, \quad r_i \geq 1, \quad a_i > 0.$$

For E we have, using $t \sim \ell(S)$, that

$$G_j^2 * [E]_t(0) \lesssim \sum_{i=1}^n a_i |B_{r_i}| \frac{1}{(r_i \ell(S))^2} \int_{[-r_i \ell(S), r_i \ell(S)]^2} G_j^2 \lesssim \|\theta\|_{L^1(\mathbb{R}^2)} 2^{2k}.$$

This implies the estimate

$$G_j^2 * [\theta \mathbf{1}_{B_1^c}]_t(0) \lesssim 2^{2k}. \quad (4.3)$$

By a translation argument, the same bound holds at any $(p, q, t) \in \Omega_{\mathcal{T}_R}$. Therefore, by (4.2) and (4.3), we have $M(G_j, \mathcal{T}_R) \lesssim 2^k$ for each j and hence

$$\sum_{S \not\subseteq H} \tilde{\Theta}_{\varphi, \psi}^{\{S\}}(G_1, G_2, G_3, G_4) \lesssim \sum_{k \leq 10} 2^{4k} \sum_{R \in \mathcal{R}_k} |R|. \quad (4.4)$$

It remains to sum up the right-hand side of the last display. Since for $R \in \mathcal{R}_k$ there is an index j such that on R we have $\mathcal{M}(G_j) > 2^{k-1}$, by maximality of the squares in \mathcal{R}_k

$$\sum_{R \in \mathcal{R}_k} |R| = \left| \bigcup_{R \in \mathcal{R}_k} R \right| \leq \sum_{j=1}^4 |\{\mathcal{M}(G_j) > 2^{k-1}\}|.$$

By the Hardy-Littlewood maximal theorem and $\|G_j\|_{L^2(\mathbb{R}^2)} \leq 1$, for each j we have $|\{\mathcal{M}(G_j) > 2^{k-1}\}| \lesssim 2^{-2k}$. Thus, (4.4) is up to a constant dominated by

$$\sum_{k \leq 10} 2^{2k} \lesssim 1.$$

This establishes the desired estimate for $S \not\subseteq H$.

Now consider the sum over all dyadic squares S contained in H . Every $S \subseteq H$ is contained in one maximal dyadic square $R \in \mathcal{R}$. Let $\mathcal{S}_{R,k}$ be the set of dyadic squares S which are k generations below $R \in \mathcal{R}$. That is, $2^k \ell(S) = \ell(R)$. We split

$$\sum_{S \subseteq H} \tilde{\Theta}_{\varphi, \psi}^{\{S\}} = \sum_{R \in \mathcal{R}} \sum_{k \geq 0} \sum_{S \in \mathcal{S}_{R,k}} \tilde{\Theta}_{\varphi, \psi}^{\{S\}}.$$

For $S \in \mathcal{S}_{R,k}$ we expand $\tilde{\Theta}_{\varphi,\psi}^{\{S\}}(G_1, G_2, G_3, G_4)$ and estimate $|\varphi^{(u)}|, |\psi^{(v)}| \lesssim \vartheta^4$ to arrive at

$$\int_{\ell(S)/2}^{\ell(S)} \int_S \int_{\mathbb{R}^4} \mathbf{F}(G_1, G_2, G_3, G_4)(x, y, x', y') [\vartheta \otimes \vartheta \otimes \vartheta \otimes \vartheta]_t(p-x, q-y, p-x', q-y') \\ \theta^2(t^{-1}(p-x, q-y)) dx dy dx' dy' dp dq \frac{dt}{t}. \quad (4.5)$$

Since G_1 is supported on the complement of $3R$, we have $|(p, q) - (x, y)| \geq \ell(R)$ for $(p, q) \in S$. We also have $\ell(R) = 2^k \ell(S) \sim 2^k t$, therefore $\theta^2(t^{-1}(p-x, q-y)) \lesssim 2^{-8k}$. Applying Lemma 5, the term (4.5) is then up to a constant dominated by

$$2^{-8k} |S| \prod_{j=1}^4 M(G_j, \{S\}).$$

Denote by R' the parent of R . For each j we have

$$M(G_j, \{S\}) \lesssim 2^k M(G_j, \{R'\}) \lesssim 2^k.$$

The last inequality follows by the same approximation argument as before and using that the averages of G_j over squares containing R' are less than 2^{10} , which is true by maximality of R . This establishes

$$\sum_{S \subseteq H} \tilde{\Theta}_{\varphi,\psi}^{\{S\}}(G_1, G_2, G_3, G_4) \lesssim \sum_{R \in \mathcal{R}} \sum_{k \geq 0} \sum_{S \in \mathcal{S}_{R,k}} 2^{-4k} |S|.$$

Since $\sum_{S \in \mathcal{S}_{R,k}} |S| \leq |R|$, the last display is estimated by

$$\sum_{R \in \mathcal{R}} |R| \sum_{k \geq 0} 2^{-4k} \lesssim |H| \lesssim 1.$$

For the second to last inequality we summed the geometric series and used disjointness of $R \in \mathcal{R}$. In the last step we used $|H| \leq 1/2$. \square

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